Volatility Estimation using GARCH Family of Models: Comparison with Option Pricing

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Abstract

Volatility estimation has been at the centre stage for Risk Management in securities market. There are various methods, which come to rescue for estimation of volatility. GARCH family of models are also in the same league and have been quite useful for estimating volatility. But post hoc there are no methods to estimate the accuracy of the forecast of the volatility estimates by various methods. Implied Volatility (IV) takes into consideration the market prices of options, and estimate the future volatility of the underlying assets. Empirically, it has been found that IV is a better future estimate of the volatility. Using current options prices in Indian derivatives market, IV has been estimated for all the permitted stocks. These IVs have been compared with volatility estimated by GARCH family of models in this paper. IGARCH (Integrated GARCH) model is giving the best results among all the other methods used in the paper.

Keywords: GARCH, Option Pricing, Implied Volatility, Risk Management, Stocks

Introduction

Although traditional research in financial economics has been concentrated on the mean of stock market returns, the later advancements in universal stock exchanges have expanded the enthusiasm for professionals, controllers and specialists towards the instability of such returns. Since the stability of the market is always at a stake, the importance to look at the volatility of the stock has got greater importance in the recent past. Basically what is volatility;it is a statistical measure of the dispersion of returns for a given security or market index. Volatility can either be measured by using the standard deviation or variance between returns from that same security or market index. The number of crashes and the size of their effects have forced all to look more carefully to the level and stationarity of volatility in time, researchers shifting their attention towards development and then improvement of econometric models able to produce accurate forecasts of such swings in returns' volatility. There are numerous models established to calculate the volatility of the stocks. Some of the most important univariate volatility models are the autoregressive conditional heteroskedastic (ARCH) model compiled by Engle (1982), generalized ARCH (GARCH) model compiled by Bollerslev (1986), the exponential GARCH (EGARCH) model of Nelson (1991), the conditional heteroskedastic autoregressive moving average (CHARMA) model obtained by Tsay (1987), the random coefficient autoregressive (RCA) model of Nicholls and Quinn (1982), and the stochastic volatility (SV) models compiled by Melino and Turnbull (1990), Taylor (1994), Harvey, Ruiz and Sentana(1994), and Jacquier, Polson, and Rossi (1994). Each model has its own strengths and weaknesses and having at hand such a large number of models, all designed to serve to the same scope, it is important to correctly distinguish between various models in order to find the one which provides the most accurate predictions. In this study, we are going to validate the efficiency of each model by comparing it with the implied volatility of each stock. This will give us a holistic picture about each model and how much we can rely upon these models and also upto which extent.

Review of Literature

The main early highlight studies to mention volatility clustering. Leptokurtosis, and leverage effect of stock return in financial market was provided by the following three studies of; Mandelbrot (1963), Fama (1965), and Black (1976).For making investment decisions it is highly essential to measure the volatility of the stock prices. Due to its significant use, several models have been proposed to capture the exact volatility of stock in the financial markets. The entire research on framing the models was started by Engle (1982), and Bollerslev (1986) who proposed the use of both ARCH and GARCH models respectively. This section will give us a brief review about the main empirical findings provided by researchers from both developed and emerging markets.

Numerous researchers found that traditional time series models that works under the fundamental supposition of steady fluctuation was not really exact in evaluating stock return movements.Thus, Engle (1982) study proposed the use of ARCH models that permits the conditional variance to change over time as a function of past errors leaving the unconditional Variance constant. ARCH model has some limitation on calculating the variance, hence Bollerslev (1986) proposed a modified form through Generalized ARCH (GARCH) to allow a longer memory and a more flexible lag structure. GARCH's main assumption regarding the conditional variance also similar to that of ARCH model where the past Variance is a linear function but the difference is the former one allows to enter the lagged conditional variance in the model itself.

Still the researchers were not satisfied with the existing models, and many extended forms of ARCH were proposed. Engle et al. (1987) in which they introduced the GARCH –M, that allows the conditional variance to be determinant of the mean. In addition their empirical findings supports that risk premium are not time invariant; rather they vary systematically.

Nelson (1991) contributed anew model through Exponential GARCH (EGARCH) which supported that variance of return was affected differently by positive and negative excess returns which broke the rigidness of GARCH model. Also the empirical findings support the negative correlation between both excess returns and stock market variance. Furthermore, depending on the previously mentioned GARCH-M model, Glosten et al. (1993) study modified the model by proposing GJR GARCH, in which their model is based on the fact that there is asymmetric response of volatility depending on the positive and negative shocks.

From that point forward progressive reviews turned out with new proposed models to the GARCH models family to conquer disadvantages of each model, such as studies by; Ding et.al (1993) proposed Asymmetric Power GARCH (APGARCH), then Zakoian (1994) threshold GARCH (TGARCH), in the investigation of Caporin and McAleer (2006) their utilized models were Dynamic Asymmetric (DAGARCH), Conditional Auto Regressive Range (CARR), and Quadratic GARCH (QGARCH) model, et cetera with more models to be applied and tried by various finacialists around the world.

Many studies were done in order to measure the correctness and efficiency of ARCH/GARCH models, few such as Hsieh(1989), Taylor (1994), Bekaert and Harvey (1997), Aggarwal et.al (1999), Brook and Burke(2003), Frimpong and Oteng (2006), and Olowe (2009); found similar conclusion that is; the best model to describe the data and measure the volatility is the GARCH (1,1). Also, they allconfirm the ability of asymmetric GARCH models in capturing asymmetry in stock return volatility.

With respect to investigations of Awartani and Corradi (2005), Miron and Tudor (2010); their systems relied on upon contrasting between different deviated models proposed already, for example, TGARCH, PGARCH, EGARCH, and GARCH-M; their principle discoveries upheld that asymmetric GARCH models assumes a crucial part in instability expectation for day by day stock return in various nations, additionally they found that EGARCH model display more wellness precision in estimation of volatility in contrast with different sorts in the asymmetric GARCH family models. Recently, there is a growing empirical researches in which their methodologies depends on applying ARCH/ GARCH models on emerging stock markets to estimate and predict volatility such as; the studies of Akgül and Sayyan (2005) and Gokbulut and Pekkaya (2014) in turkey, Goudarzi and Ramanarayanan (2011) in India. Their main findings were the occurrence of non-normality, volatility clusters, negative skewness, leptokurtosis for data gathered from emerging economies; in addition, the best fit model for the data is GARCH (1, 1). Also, Gokbulut and Pekkaya (2014) supported that the CGARCH and TGARCH appear to be superior in modeling volatility.

Many researchers are keenly involved in comparing the different ARCH/GARCH family models on the data gathered from different economies, so that we can figure out the best suited model for that particular type of economic condition. Although there are many studies around the world that tested the ARCH/GARCH family models in their capital markets, few studies were found in Jordan concerning this issue. Rousan and Al Khouri (2005) study investigated the volatility for Amman Stock Exchange (ASE) for the period during (1992-2004), depending on daily observations for the general index of the exchange; and their results support that ARCH/GARCH models can provide good approximation for capturing the characteristics of ASE. In addition the study applied multiple asymmetric models to track the leverage effect and found that the exchange is symmetric; hence, good and bad news has the same magnitude. Alraimony and Nader (2012) measured the volatility and the effect of macroeconomics on it by applying ARCH/ GARCH, and their findings were that the ARCH was found statistically significant. On the other hand, GARCH was found statistically insignificant during the period (1991-2010). Our main objective is to calculate the implied volatility of all the option trading companies in India, and comparing it with the volatility calculated using the GARCH, IGARCH, EGARCH and CGARCH models using statistical t-test to test the significance of each models.

Theoretical Background

ARCH

ARCH model assumes that variance of tomorrow's return is an equally weighted average of the squared residuals of the last available data, it could be in terms of number of days or months or even years. The assumption of equal weights seems unattractive as one would think that the more recent events would be more relevant and therefore should have higher weights. Furthermore the assumption of zero weights for observations more than one month old, is also unattractive. The ARCH model was to estimate the weights of parameters. Thus the model allowed the data to determine the best weights to use in forecasting the variance.

$$\sigma_t{}^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i \varepsilon^2{}_{t-i}$$

Where VL is the long term variance and the weights is equal to unity.

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

GARCH

A useful generalization of this model is the GARCH parameterization introduced by Bollerslev(1986). This model is also a weighted average of past squared residuals but it has declining weights which never go completely to zero. It gives parsimonious models which are easy to estimate and even in its simplest form, has proven surprisingly successful in predicting conditional variances. The most widely used GARCH specification, asserts that the best predictor of the variance in the next period is a weighted average of the long run average variance, the variance predicted for this period and the new information this period which is the most recent squared residual. Such an updating rule is a simple description of adaptive or learning behavior and can be thought of as Bayesian updating.

GARCH(1,1) model is given as follows,

$$\sigma_t{}^2 = \gamma V_L + \alpha \varepsilon^2{}_{t-1} + \beta \sigma^2{}_{t-1}$$

Since α, β and γ are weights there sum is unity and also ω = γV_L

$$\alpha + \beta + \gamma = 1$$

Although this model is directly set up to forecast for just one period, it turns out that based on the one period forecast a two period forecast can be made. Ultimately by repeating this step, long horizon forecasts can be constructed. The likelihood function provides a systematic way to adjust the parameters β, a , ? to give the best fit.

The GARCH (1,1) model can be generalized to a GARCH(p,q) model; that is, a model with additional lag terms. Such higher order models are often useful when a long span of data is used, like several decades of daily data or a year of hourly data. With additional lags, such models allow both fast and slow decay of information.

IGARCH

According to GARCH model, the covariance is stationary

$$\alpha(1) + \beta(1) < 1$$

But the stationarity does not require such a stringent restriction, i.e the unconditional variance does not depend on t. practically, the covariance is not stationary,

$$\alpha(1) + \beta(1) = 1$$

This modified model is termed as Integrated GARCH model (iGARCH). This model was developed by Engel and Bollerslev (1986).

$$\sigma_t{}^2 = \alpha_0 + (1 - \beta_1)\varepsilon^2{}_{t-1} + \beta_1 \sigma^2{}_{t-1}$$

However we may suspect that IGARCH is more a product of omitted structural breaks than the result of true IGARCH behavior.

EGARCH

The negative correlation between stock returns and changes in returns volatility, i.e. volatility tends to rise in response to "bad news", (excess returns lower than expected) and to fall in response to "good news" (excess returns higher than expected). GARCH models, however, assume that only the magnitude and not the positivity or negativity of unanticipated excess returns determines feature $\sigma^2{}_t$

$$\ln(\sigma^2) = \omega + \sum_{i=1}^{p} \beta_i \ln(\sigma^2{}_{t-i}) + \sum_{i=1}^{q} \alpha_i \left[\emptyset z_{t-i} + \varphi(|z_{t-i}| - E|z_{t-i}|) \right]$$

In the EGARCH model, the conditional variance, σ_t, is an asymmetric function of lagged disturbances.

CGARCH

The component GARCH model of Engle and Lee (1999) was designed to better account for long-run volatility dependencies. Rewriting GARCH (1, 1) model as follows

$$(\sigma^2{}_t - \sigma^2) = \alpha\,(\varepsilon^2{}_{t-1} - \sigma^2) + \beta\,(\sigma^2{}_{t-1} - \sigma^2)$$

Where $\sigma^2 = \frac{\omega}{(1-\alpha-\beta)}$ refers to the unconditional variance, the CGARCH model obtained by relaxing the assumption of a constant s 2. Specifically,

$$(\sigma^2{}_t - \xi^2{}_t) = \alpha\left(\varepsilon^2{}_{t-1} - \xi^2{}_{t-1}\right) + \beta\,(\sigma^2{}_{t-1} - \xi^2{}_{t-1})$$

With the corresponding long-run variance parameterized by the separate equation,

$$\xi^2{}_t = \omega + \rho(\xi^2{}_{t-1}) + \psi\,(\varepsilon^2{}_{t-1} - \sigma^2{}_{t-1})$$

T- Test

A t-test is an analysis of two population means through the use of statistical examination; a t-test with two samples is commonly used with small sample sizes, testing the difference between the samples when the variances of two normal distributions are not known. A t-test looks at the t-statistic, the t-distribution and degrees of freedom to determine the probability of difference between populations; the test statistic in the test is known as the t-statistic. To conduct a test with three or more variables, an analysis of variance (ANOVA) must be used.

Assumptions:

As a parametric procedure, the t-test makes several assumptions. Although t-tests are quite robust, it is good practice to evaluate the degree of deviation from these assumptions in order to assess the quality of the results. The t-test has four main assumptions:

- The dependent variable must be continuous (interval/ratio).
- The observations are independent of one another.
- The dependent variable should be approximately normally distributed.
- The dependent variable should not contain any outliers.

The procedure for one sample t-test are first they calculate the mean and the standard deviation of the sample data that has been collocated and then they find the test statistic using the below formula

$$t = \frac{\overline{x} - \mu}{\sqrt{s^2/n}}$$

Where x the sample mean, s2 is the sample variance, μ is the specified population mean and n is the sample size.

Once the t statistic is calculated, the probability of observing the test statistic under the null hypothesis is calculated. Statistical significance is determined by looking at the p-value. The p-value gives the probability of observing the test results under the null hypothesis. If the assumptions are true, smaller p-values indicate a result that is less likely to occur by chance. This also indicates decreased support for the null hypothesis, although this possibility can never be ruled out completely. The cutoff value at which statistical significance is claimed is decided on by the researcher but usually a value of .05 or less is chosen, ensuring approximately 95% confidence in the results.

We have used t-test in this paper to statistically test if there is any significant difference in the volatility calculated using the various volatility models and the implied volatility.

Data And Methodology:

Data:

The prices of at-the-money call option for each of the 170 companies listed in the Futures & Options (F&O) segment of the NSE are collected from the NSE website. Further the spot prices of each underlying stock on the particular day and the strike price of each option are also taken. The period of collection of the data was 15th-18th March, 2017.

Additionally, the historical weekly closing prices of the 170 companies were collected for the last 3 years using Prowess IQ. This data is used to find the volatility of the stocks using the various volatility models.

Methodology:

Calculation of Implied volatility:

The implied volatility of a stock is simply the volatility at which the market price of the option equals the price determined using the Black and Scholes option pricing model, when all other parameters are constant. The implied volatility gives an intuitive understanding of the market expectation of the future price movements. The implied volatility of all the 170 companies are determined using Derivagem.

Determination of volatility using different volatility models:

The volatility of a stock is given by the standard deviation of the historical returns. One of the main problems in financial time series like stock price returns is that the volatility is not considered to be constant with respect to time i.e. the time series is considered to be heteroskedastic. Therefore the normal method of calculating the standard deviation from historical data may not be a good estimate of the future volatility.

To overcome the problem of heteroskedacity, GARCH models are used in this study. Different GARCH models are used in this study as each of the models take care of some of the inherent properties of time series such as volatility clustering, mean reversion and the leverage effects. The volatility models used in this study are given below:

- GARCH (1,1)
- Exponential GARCH or E-GARCH
- Threshold GARCH or T-GARCH
- Integrated GARCH or I-GARCH
- Component GARCH or C-GARCH

Using the historical data, all the above models are estimated for each of the 170 stocks taken for study. E-views software is used in the modelling of the time series. Therefore using each of the different models, different values of volatility can be estimated for each stock. Therefore, a series of 6 different volatilities including the implied volatility can be obtained for each of the 170 stocks.

Comparison of the different volatilities:

The volatility from different models are compared to the implied volatility of the stocks. The objective is to identify which volatility model closely explains the expectation of the market participants about the future volatility. This also could be used to understand more effectively the actual properties of the volatility of the time series.

A one sample t test can be used to compare the different volatility models with the implied volatility. The null hypothesis is that the two volatilities have no significant difference between them and therefore are equal. If the hypothesis is not rejected we may conclude that the particular model must reflect the market expected volatility.

The t test is done by taking the difference of two volatility models at a time and testing whether that difference is equal to Zero or not. If the difference is significantly different from zero, then it means that the two models produce volatilities which are different from each other. The p-values are used to indicate whether the hypothesis can be rejected or not.

Results:

Calculation of volatilities using the different methods:

The volatilities of all the stock prices are determined from all the above mentioned 5 methods using Eviews software. The implied volatility of the stocks are also calculated using derivagem. Then the differences in each pair of volatility is calculated. The descriptive statistics of the distribution of all the models is given in the table 1.

Table 1: Descriptive statistics for the distribution of the various volatility models.

	Mean	Median	Mode	Standard Deviation	Total N
IV vs. GARCH	-0.031	-0.025	-0.251	0.077	170
IV vs. EGARCH	-0.029	-0.019	-0.326	0.087	170
IV vs. IGARCH	-0.011	-0.002	-0.869	0.098	170
IV vs. CGARCH	-0.030	-0.016	-0.617	0.102	170
IV vs. TARCH	-0.026	-0.028	-0.238	0.073	170
GARCH vs. EGARCH	0.002	0.003	-0.287	0.059	170
GARCH vs. IGARCH	0.020	0.025	-0.765	0.080	170
GARCH vs. CGARCH	0.000	0.002	-0.512	0.060	170
GARCH vs. TARCH	0.005	0.000	-0.147	0.045	170
EGARCH vs. IGARCH	0.018	0.021	-0.776	0.097	170
EGARCH vs. CGARCH	-0.001	0.000	-0.523	0.079	170
EGARCH vs. TARCH	0.003	-0.001	-0.136	0.056	170
IGARCH vs. CGARCH	-0.019	-0.016	-0.324	0.072	170
IGARCH vs. TGARCH	-0.015	-0.021	-0.175	0.090	170
CGARCH vs. TARCH	0.004	-0.003	-0.279	0.076	170

Comparison of the volatilities determined from various methods:

The results of the one sample t test are given in the table 1. The table consists of the p-values obtained by testing the difference between each of the two models. The table is symmetrical so it can be read either horizontally or vertically. For example, the p-value obtained from the one sample t-test between the GARCH and EGARCH is 0.724.

Table 2: P-values obtained from one sample t tests.

	Implied volatility	GARCH	EGARCH	IGARCH	CGARCH	TGARCH
Implied volatility		0.000	0.000	0.151	0.000	0.000
GARCH	0.000		0.724	0.002	0.966	0.179
EGARCH	0.000	0.724		0.017	0.818	0.476
IGARCH	0.151	0.002	0.017		0.001	0.032
CGARCH	0.000	0.966	0.818	0.001		0.442
TGARCH	0.000	0.179	0.476	0.032	0.442	

Source: SPSS output

From the above table, it can be seen that all the models except IGARCH are significantly different from the implied volatility of the stocks even at 1% level. This enables us to understand the volatility obtained from the I-GARCH model is more helpful in understanding about the market sentiment than the other models.

Another notable observation is that the models GARCH, E-GARCH, C-GARCH, T-GARCH are not significantly different one another. But all of the models are significantly different from the I-GARCH model. This indicates that the I-GARCH model accounts for certain unique properties of the time series which also explains the market sentiment better than all the other models.

The I-GARCH is a special case of GARCH model in which the sum of the weights given to the previous residual and the previous variance term is made equal to one. The I-GARCH model does not take care of the long term variance term which is present in other GARCH models. This means that the concept of mean reversion is not taken care by the I-GARCH model. If the I-GARCH model does explain the market sentiment as concluded from our study, it could mean that the market also does not expect to be reverted back to the long term mean value. In a way, this result tends to violate the general hypothesis that markets are efficient.

Conclusion:

The estimation of volatility of stock returns plays a major role in various areas of finance such as pricing of derivatives, calculation of VaR and other risk management tools as well as in the forecasting of stock prices. Several studies have been done to understand the market expectation of future prices using the volatility. The implied volatility of the stock is obtained by the substituting the market price of an option in the Black-Scholes Model. The implied volatility, though it gives a theoretical understanding of the market sentiment, does not help in understanding the property of the return series with respect to time. This study is an attempt to identify the appropriate model which can help investors, analysts, managers and other researchers in understanding the properties of the stock returns. Each model in the family of GARCH model takes into consideration a unique property of the time series. A comparative analysis between the different models using one sample t-test shows that only the I-GARCH model produces statistically similar results to the implied volatility. The properties which are accounted in the I-GARCH model are more likely to explain the properties of returns than other models. This result can be interpreted that the market believes that any change that has happened in the past is more likely to be sustained in the future, violating the concept of mean reversion and the efficient market hypothesis.

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